Finite-difference method for boundary-value problems

1. Given a linear ordinary differential equation (LODE) and two boundary conditions, converting the LODE into a finite-difference equations allows us to define a system of $n - 1$ linear equations in $n - 1$ unknowns. This defines a matrix and a vector. What simplifications to the matrix are there if the coefficients of $u(x)$ and its derivatives are all constant coefficients?

Answer: All entries on the diagonal, the super-diagonal, and the sub-diagonal are equal, respectively.

2. What is the conditions of the index of an entries of a matrix $a_{i,j}$ if

a. the entry is on the diagonal,
b. the entry is on the super-diagonal, or
c. the entry is on the sub-diagonal?

Answer: $i = j$ or $i - j = 0$; $i = j - 1$, $i + 1 = j$, or $i - j = -1$; and $i = j + 1$, $i - 1 = j$, or $i - j = 1$.

3. Convert this LODE into a finite-difference equation:

$$u^{(2)}(x) - 0.5u^{(1)}(x) - 2u(x) = 0.$$

Answer: $(2 + 0.5h)u(x - h) + (-4 - 4h^2)u(x) + (2 - 0.5h)u(x + h) = 0$.

4. What changes to the previous answer if the forcing function is some given function $g(x)$?

Answer: The right-hand side, as opposed to being 0, is now $2g(x)h^2$.

5. Suppose the homogenous LODE has the boundary conditions $u(1) = 2$ and $u(3) = 4$ and we are dividing the interval into four sub-intervals. What is the matrix and vector that defines the system of linear equations?

Answer: $$\begin{pmatrix} -5 & 1.75 & \\ 2.25 & -5 & 1.75 \\ & 2.25 & -5 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -4.5 \\ 0 \\ -7 \end{pmatrix}$$

6. Suppose that you have the same boundary conditions, but the forcing function is now x^2; how does this change the system of linear equations?

Answer: $$\begin{pmatrix} -5 & 1.75 & \\ 2.25 & -5 & 1.75 \\ & 2.25 & -5 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -3.375 \\ 2 \\ -3.875 \end{pmatrix}$$

7. How does the above matrix change if we double the number of points?

Answer: $$\begin{pmatrix} -4.25 & 1.875 & & & & & \\ 2.125 & -4.25 & 1.875 & & & & \\ & 2.125 & -4.25 & 1.875 & & & \\ & & 2.125 & -4.25 & 1.875 & & \\ & & & 2.125 & -4.25 & 1.875 & \\ & & & & 2.125 & -4.25 & 1.875 \\ & & & & & 2.125 & -4.25 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix} = \begin{pmatrix} -4.0546785 \\ 0.28125 \\ 0.3828125 \\ 0.5 \\ 0.6328125 \\ 0.78125 \\ -6.5546875 \end{pmatrix}$$

8. Suppose you have the BVP $u^{(2)}(x) - 2u^{(1)}(x) - xu(x) = 1$ with $u(0) = 1$ and $u(5) = 2$. What is the matrix and vector that define the system of linear equations if $n = 5$?

Answer: $$\begin{pmatrix} -6 & & & \\ 4 & -8 & & \\ & 4 & -10 & \\ & & 4 & -12 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$$

9. What are the matrix and vector that defines the systems of linear equations for the previous question if $n = 10$?

Answer: $$\begin{pmatrix} -4.25 & 1 & & & & & & & \\ 3 & -4.5 & 1 & & & & & & \\ & 3 & -4.75 & 1 & & & & & \\ & & 3 & -5 & 1 & & & & \\ & & & 3 & -5.25 & 1 & & & \\ & & & & 3 & -5.5 & 1 & & \\ & & & & & 3 & -5.75 & 1 & \\ & & & & & & 3 & -6 & 1 \\ & & & & & & & 3 & -6.25 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} -2.5 \\ 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \\ -1.5 \end{pmatrix}$$

10. When we had Neumann conditions, that specified the slopes at the end points as opposed to the values of the solutions, we used one of the following two approximations of the derivative:

$$\frac{-3u_0 + 4u_1 - u_2}{2h} \text{ or } \frac{u_{n-2} - 4u_{n-1} + 3u_n}{2h}$$

instead of the easier approximations

$$\frac{-u_0 + u_1}{h} \text{ or } \frac{u_n - u_{n-1}}{h}.$$

What was the primary reason for this?

Answer: The other divided-difference approximations of both the derivative and second derivative are both O(h^2), but mixing O(h) approximations (the second above) results in only O(h) accuracy, while using the first two approximations at the end points results in O(h^2) approximations overall.

11. Suppose you have the BVP $u^{(2)}(x) - 5u^{(1)}(x) - 4u(x) = 1$ with $u(0) = 1$ and $u^{(1)}(5) = -1$. What is the matrix and vector that define the system of linear equations if $n = 5$?

Answer: $$\begin{pmatrix} -12 & -3 & & \\ 7 & -12 & -3 & \\ & 7 & -12 & -3 \\ & & 8 & -16 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 2 \\ 0 \end{pmatrix}$$

11. Suppose you have the BVP $u^{(2)}(x) - 4u^{(1)}(x) - 4u(x) = 1$ with $u^{(1)}(0) = 1$ and $u(5) = -1$. What is the matrix and vector that define the system of linear equations if $n = 5$?

Answer: $$\begin{pmatrix} -4 & -4 & & \\ 6 & -12 & -2 & \\ & 6 & -12 & -2 \\ & & 6 & -12 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 2 \\ 0 \end{pmatrix}$$

12. We have the values of u_k, but we don't have the derivatives at the points x_k. How would you approximate the derivative at x_k?

Answer: Use the formula $\frac{u_{k+1} - u_{k-1}}{2h}$; after all, this is the approximation we used to create the finite-difference approximation.

n and see what the derivative is at $u(5)$.